The framed discs operad is cyclic

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Abstract

The operad of framed little discs is shown to be equivalent to a cyclic operad. This answers a conjecture of Salvatore in the affirmative, posed at the workshop 'Knots and Operads in Roma,' at Università di Roma "La Sapienza" in July of 2006.

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1 Statement and proof

Both the author [1] and Paolo Salvatore [8] have constructed actions of the operad of little 2-cubes on the 'framed' long knot spaces $\mathrm{EC}(1, D^n)$ defined in [1] which extend the homotopy-associative product given by the connect-sum operation. Computations of the homotopy-type [2] and homology [3] of $\mathrm{EC}(1, D^2)$ suggest in several ways (some more vague than others) that the spaces $\mathrm{EC}(1, D^n)$ may admit an action of the operad of framed little 2-discs. A concrete example: provided such an action exists, it would allow for a more compact and 'operadic' description of the homotopy-type of $\mathrm{EC}(1, D^2)$ than the one given in [1, 2]. The author has made several attempts to construct an action of the operad of framed 2-discs $\mathfrak{f}\mathcal{D}_2$ on $\mathrm{EC}(1, D^n)$ in the spirit of 'Little cubes and long knots' [1], as of yet without success. Salvatore constructs the spaces $\mathrm{EC}(1, D^n)$ for $n \geq 3$ as the totalisation of a multiplicative operad, further developing the work of Sinha [11]. The operad Salvatore uses is equivalent to the operad of framed little $(n+1)$-discs $\mathfrak{f}\mathcal{D}_{n+1}$. Theorems of McClure and Smith imply that the totalisation of a multiplicative operad is a module over the operad of 2-cubes [7]. Motivated by these constructions, Salvatore asked if the operad of framed discs is cyclic. The geometric techniques used in the author's failed attempts to construct an action of $\mathfrak{f}\mathcal{D}_2$ directly on $\mathrm{EC}(1, D^n)$ turn out to answer his question. In Proposition 1.1, $\mathfrak{f}\mathcal{D}_n$ is shown to be equivalent in the sense of Fiedorowicz [4] to the operad of 'conformal n-balls' $\mathcal{CB}_n$. In Theorem 1.2, $\mathcal{CB}_n$ is shown to be cyclic.

This paper follows the conventions of the book 'Operads in Algebra, Topology and Physics' by Markl, Shnider and Stasheff [6]. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. Consider $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots$. Let $p_+ = (0, \cdots, 0, 1) \in S^n$ and $p_- = -p_+$ be the north and south poles of S^n respectively. Let $D^n \subset S^n$ be the 'southern' hemisphere in S^n, meaning $\partial D^n = S^{n-1}$ and $p_- \in D^n$. Given a point $v \in S^n$, identify the tangent space $T_{-v}S^n$ with $S^n \setminus \{v\}$ via stereographic projection from v. Since stereographic projection is a conformal diffeomorphism, one can consider a conformal affine-linear transformation of $T_{-v}S^n$ to be a conformal diffeomorphism of $S^n \setminus \{v\}$ and further extend it to a conformal diffeomorphism of S^n. Let $\mathrm{CDiff}(S^n)$ be the group of orientation-preserving conformal diffeomorphisms of S^n. Let $\mathrm{CDiff}(S^n, v) \subset \mathrm{CDiff}(S^n)$ be the subgroup corresponding to the conformal affine-linear transformations of $T_{-v}S^n$. Let $\mathrm{Trans}(v) \subset \mathrm{CDiff}(S^n, v)$ be the subgroup consisting only of translations in $T_{-v}S^n$. Let SO_{n+1} denote the group of orientation-preserving isometries of S^n. Let $\mathrm{SO}_n(v)$ be defined as $\mathrm{SO}_{n+1} \cap \mathrm{CDiff}(S^n, v)$. Define $\mathrm{Scale}(v) \subset \mathrm{CDiff}(S^n, v)$ to be the scalar multiples of the identity on $T_{-v}S^n$.

For $n \geq 2$, $\mathrm{CDiff}(S^n)$ is known to be a $\binom{n+1}{2}$-dimensional Lie group, isomorphic to the group of hyperbolic isometries of hyperbolic $(n+1)$-space. The isomorphism is given by restriction to the projectivised light-cone of the Minkowski model. Stated another way, for any $v \in S^n$, $\mathrm{CDiff}(S^n)$ is generated by the subgroups SO_{n+1}, $\mathrm{Trans}(v)$ and $\mathrm{Scale}(v)$. Moreover, the stabiliser of $v \in S^n$ under the action of $\mathrm{CDiff}(S^n)$ is $\mathrm{CDiff}(S^n, v)$. For details, see Schoen and Yau's book [10] §VI Theorem 1.1.

Let $\pi : S^n \to S^n$ denote rotation by 180-degrees around $S^{n-2} \subset S^n$. A $(j+1)$-tuple $(\pi, f_1, \cdots, f_j) \in \mathrm{CDiff}(S^n)^{j+1}$ will be called j little conformal n-balls if $f_i(D^n) \subset D^n$ for all i, and if the interior of $f_i(D^n)$ is disjoint from $f_k(D^n)$ for all $i \neq k$. The set of all such $(j+1)$-tuples is denoted $\mathcal{CB}_n(j)$. There are maps $\mathcal{CB}_n(i) \times (\mathcal{CB}_n(j_1) \times \cdots \mathcal{CB}_n(j_i)) \to \mathcal{CB}_n(j_1 + \cdots + j_i)$ defined by sending $((\pi, f_1, \cdots, f_i), (\pi, g_{11}, \cdots, g_{1j_1}), \cdots, (\pi, g_{i1}, \cdots, g_{ij_i}))$ to $(\pi, f_1 g_{11}, \cdots, f_1 g_{1j_1}, \ \cdots, f_i g_{i1} \cdots f_i g_{ij_i})$. Consider $\mathcal{CB}_n(j)$ to be a subspace of $\mathrm{CDiff}(S^n)^{\{0,1,\cdots,j\}}$. Then there is a natural right action of Σ_j on $\mathcal{CB}_n(j)$. The above maps together with the actions of Σ_j for $j \in \{1, 2, \cdots\}$ endow $\mathcal{CB}_n$ with the

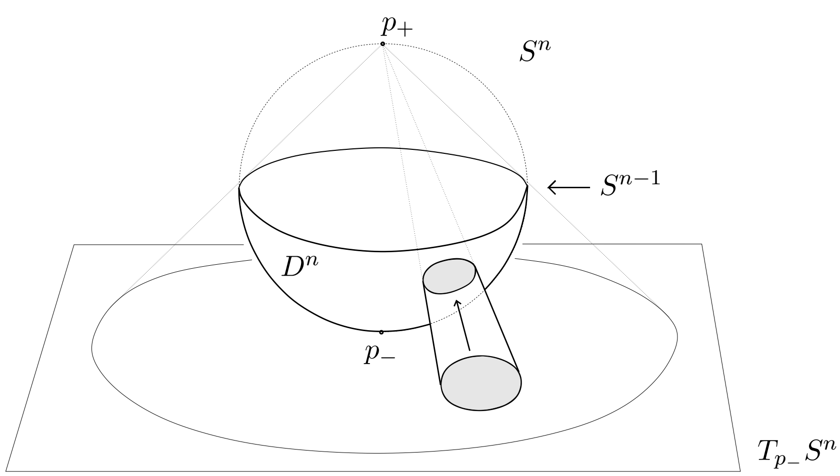
structure of an operad, hereby named the operad of conformal n-balls. Observe that for $n \geq 2$ the operad of framed little n-discs $\mathfrak{f}\mathcal{D}_n$ is the suboperad of $\mathcal{CB}_n$ such that the north pole is fixed by each disc: $f_i(p_+) = p_+$ for all $i \in \{1, \cdots, j\}$.

Proposition 1.1 *There is a deformation-retraction of $\mathcal{CB}_n$ to the suboperad $\mathfrak{f}\mathcal{D}_n$. Moreover, the inclusions $\mathfrak{f}\mathcal{D}_n(j) \to \mathcal{CB}_n(j)$ are Σ_j-equivariant homotopy-equivalences for all $j \in \mathbb{N}$.*

Proof To construct the homotopy-inverse $\mathcal{CB}_n \to \mathfrak{f}\mathcal{D}_n$, consider $f \in \mathrm{CDiff}(S^n)$ such that $f(D^n) \subset D^n$. f can be written uniquely as a composite $f = Y \circ S \circ M \circ Q$ where $Y \in \mathrm{Trans}(p_+)$, $S \in \mathrm{Scale}(p_+)$, $M \in \mathrm{SO}_n(p_+)$, $Q \in \mathrm{Trans}(p_-)$ and $f_i(p_-) = Q(p_-)$:

- Q^{-1} is translation by $f^{-1}(p_+) \in T_{p_+}S^n$.
- Y^{-1} is translation by $(f \circ Q^{-1})(p_-) \in T_{p_-}S^n$.
- $Y^{-1} \circ f \circ Q^{-1}$ fixes both p_+ and p_-, so it is an element of $\mathrm{Scale}(p_+)\mathrm{SO}_n(p_+) = \mathrm{Scale}(p_+) \times \mathrm{SO}_n(p_+)$, and so there is a unique solution to the equation $S \circ M = Y^{-1} \circ f \circ Q^{-1}$.

Given $(\pi, f_1, \cdots, f_j) \in \mathcal{CB}_n(j)$, let $f_i = Y_i \circ S_i \circ M_i \circ Q_i$ be as above. Let $Q_i(t)$ for $t \in [0,1]$ be the straight line in $\mathrm{Trans}(p_-)$ from Q_i to the identity. Notice that $Y_i \circ S_i \circ M_i \circ Q_i(t)(p_-) = f_i(p_-)$ for all i and t. For all $t \in [0,1]$ there exists $r(t) \in (0,1]$ such that if $R_t \in \mathrm{Scale}(p_+)$ denotes scaling by $r(t)$ in $T_{p_-}S^n$, then $(\pi, Y_1 \circ S_1 \circ R_t \circ M_1 \circ Q_1(t), \cdots, Y_j \circ S_j \circ R_t \circ M_j \circ Q_j(t)) \in \mathcal{CB}_n(j)$. For each $t \in [0,1]$ choose $r(t)$ to be the maximal possible such element. This makes $r(t)$ a continuous function of t. Observe $(\pi, Y_1 \circ S_1 \circ R_1 \circ M_1 \circ Q_1(1), \cdots, Y_j \circ S_j \circ R_1 \circ M_j \circ Q_j(1)) = (\pi, Y_1 \circ S_1 \circ R_1 \circ M_1, \cdots, Y_j \circ S_j \circ R_1 \circ M_j) \in \mathfrak{f}\mathcal{D}_n$. □



The inclusion $\mathfrak{f}\mathcal{D}_n \to \mathcal{CB}_n$

One of the main roles of the operad of framed discs $\mathfrak{f}\mathcal{D}_n$ is that it detects when a space X has the homotopy-type of an n-fold loop space $\Omega^n Y$ where Y is an SO_n-space [9]. By the work of Fiedorowicz [4], the above proposition implies that $\mathcal{CB}_n$ also detects n-fold loop spaces over SO_n-spaces.

Theorem 1.2 *For $n \geq 2$, $\mathcal{CB}_n$ is a cyclic operad.*

Proof We define a right Σ_j^+-action on $\mathcal{CB}_n(j)$ which extends the Σ_j action. Here Σ_j^+ is the permutation group of $\{0,1,\cdots,j\}$ and Σ_j is the stabiliser of 0. Given $\sigma \in \Sigma_j^+$ and $f \in \mathcal{CB}_n(j)$, let $f.\sigma = \pi f_{\sigma(0)}^{-1} f\sigma$. Juxtaposition is interpreted as composition of functions, where $f \in \mathcal{CB}_n(j)$ is considered to be a an element of $\mathrm{CDiff}(S^n)^{\{0,1,\cdots,j\}}$. More explicitly, $\sigma.(\pi, f_1, \cdots, f_j) = (\pi, \pi f_{\sigma(0)}^{-1} f_{\sigma(1)}, \cdots, \pi f_{\sigma(0)}^{-1} f_{\sigma(j)})$ where we interpret f_0 as π, if it appears. This is a right action of Σ_j^+, which can be verified by a quick calculation:

$$\begin{aligned}(f.\sigma).\epsilon &= \left(\pi f_{\sigma(0)}^{-1} f\sigma\right).\epsilon = \pi f_{\sigma\epsilon(0)}^{-1} f_{\sigma(0)}\pi\left(\pi f_{\sigma(0)}^{-1} f\sigma\epsilon\right)\\ &= \pi f_{\sigma\epsilon(0)}^{-1} f\sigma\epsilon = f.(\sigma\epsilon)\end{aligned}$$

The three axioms of a cyclic operad are verified below.

Axiom (i)

$$\begin{tikzcd} \mathcal{CB}_n(1) \arrow[rr, "\tau_1"] & & \mathcal{CB}_n(1) \\ & 1 \arrow[ul, "\eta"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

states that the transposition $(01) = \tau_1$ fixes the identity element of $\mathcal{CB}_n(1)$, which can be checked: $(\pi, I).\tau_1 = (\pi I^{-1} I, \pi I^{-1}\pi) = (\pi, I)$.

Axiom (ii) states that the diagram below commutes:

$$\begin{tikzcd} \mathcal{CB}_n(i) \times \mathcal{CB}_n(j) \arrow[r, "\circ_1"] \arrow[d, "\tau_i \times \tau_j"] & \mathcal{CB}_n(i+j-1) \arrow[dd, "\tau_{i+j-1}"] \\ \mathcal{CB}_n(i) \times \mathcal{CB}_n(j) \arrow[d, "s"] & \\ \mathcal{CB}_n(j) \times \mathcal{CB}_n(i) \arrow[r, "\circ_i"] & \mathcal{CB}_n(i+j-1) \end{tikzcd}$$

which we check by going around the diagram two ways, first the clockwise direction.

$$\begin{aligned}(f \circ_1 g).\tau_{i+j-1} &= (\pi, f_1g_1, \cdots, f_1g_j, f_2, \cdots, f_i).\tau_{i+j-1}\\ &= \pi g_1^{-1} f_1^{-1}(f_1g_1, f_1g_2, \cdots, f_1g_j, f_2, \cdots, f_i, \pi)\\ &= (\pi, \pi g_1^{-1}g_2, \cdots, \pi g_1^{-1}g_j, \pi g_1^{-1}f_1^{-1}f_2, \cdots, \pi g_1^{-1}f_1^{-1}f_i, \pi g_1^{-1}f_1^{-1}\pi)\end{aligned}$$

and the counter-clockwise direction

$$\begin{aligned}\circ_j(s(f.\tau_i, g.\tau_j))) &= (g.\tau_j) \circ_j (f.\tau_i)\\ &= (\pi, \pi g_1^{-1}g_2, \cdots, \pi g_1^{-1}g_j, \pi g_1^{-1}\pi) \circ_j (\pi, \pi f_1^{-1}f_2, \cdots, \pi f_1^{-1}f_i, \pi f_1^{-1}\pi)\\ &= (\pi, \pi g_1^{-1}g_2, \cdots, \pi g_1^{-1}g_j, \pi g_1^{-1}\pi^2 f_1^{-1}f_2, \cdots, \pi g_1^{-1}\pi^2 f_1^{-1}f_i, \pi g_1^{-1}\pi^2 f_1^{-1}\pi)\\ &= (\pi, \pi g_1^{-1}g_2, \cdots, \pi g_1^{-1}g_j, \pi g_1^{-1}f_1^{-1}f_2, \cdots, \pi g_1^{-1}f_1^{-1}f_i, \pi g_1^{-1}f_1^{-1}\pi)\end{aligned}$$

Axiom (iii) states that the diagram below commutes:

$$\begin{tikzcd} \mathcal{CB}_n(i) \times \mathcal{CB}_n(j) \arrow[r, "\circ_i"] \arrow[d, "\tau_i \times 1"] & \mathcal{CB}_n(i+j-1) \arrow[d, "\tau_{i+j-1}"] \\ \mathcal{CB}_n(i) \times \mathcal{CB}_n(j) \arrow[r, "\circ_{i-1}"] & \mathcal{CB}_n(i+j-1) \end{tikzcd}$$

which we check by going around the diagram both ways, first clockwise.

$$\begin{aligned}(f \circ_k g).\tau_{i+j-1} &= (\pi, f_1, \cdots, f_{k-1}, f_k g_1, \cdots, f_k g_j, f_{k+1}, \cdots, f_i).\tau_{i+j-1} \\ &= (\pi, \pi f_1^{-1} f_2, \cdots, \pi f_1^{-1} f_{k-1}, \pi f_1^{-1} f_k g_1, \cdots, \pi f_1^{-1} f_k g_j, \pi f_1^{-1} f_{k+1}, \cdots, \pi f_1^{-1} f_i, \pi f_1^{-1} \pi)\end{aligned}$$

and counter-clockwise

$$\begin{aligned}f.\tau_i \circ_{k-1} g &= (\pi, \pi f_1^{-1} f_2, \cdots, \pi f_1^{-1} f_i, \pi f_1^{-1} \pi) \circ_{k-1} (g_1, \cdots, g_j) \\ &= (\pi, \pi f_1^{-1} f_2, \cdots, \pi f_1^{-1} f_{k-1}, \pi f_1^{-1} f_k g_1, \cdots, \pi f_1^{-1} f_k g_j, \pi f_1^{-1} f_{k+1}, \cdots, \pi f_1^{-1} f_i, \pi f_1^{-1} \pi)\end{aligned}$$

□

The operad of little n-cubes (or unframed little n-discs) is known not to be cyclic provided $n \geq 2$ is even, since its homology is not cyclic. A proof appears in Proposition 3.18 of [5].

$\mathfrak{f}\mathcal{D}_1$ and $\mathfrak{f}\mathcal{D}_2$ were already known to be cyclic before the publication of this article [5, 6]. The proof above can be adapted to give a coherent proof that $\mathcal{CB}_n$ is cyclic for all $n \geq 1$, provided one defines $\mathrm{CDiff}(S^1)$ to be the group of orientation-preserving isometries of hyperbolic 2-space, restricted to the circle at infinity. One defines $\pi : S^1 \to S^1$ to be rotation by 180 degrees about the centre of the circle, and the proof proceeds as above.

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